What if the Model Doesn't Work?

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1. Introduction

THE modelling process is often illustrated in the form of a simple flowchart – as in Raggett's article in the inaugural issue of this journal.[1] Actions have to be taken and decisions made concerning the adequacy of the model, the difficulty of obtaining a mathematical solution and the reality of any such solution. Most of these processes are quite alien to many students brought up on conventional A-level syllabuses where problems are usually posed with just the right amount of information to set up a model, the required mathematical technique is more or less implied by the wording of the problem and the model has a nice, straightforward solution. In a foundation course we try to pave the way for further mathematical developments by looking at examples where an uncritical application of techniques may lead to anomalies. This article considers one such example.

2. The problem

The problem is to pass a long bookcase 1ft wide through a doorway 4ft wide and 7ft high. By tilting the bookcase it is possible that it can pass through the opening even if it is more than 7ft tall – what is the maximum height of a bookcase that will pass through the door? In fact I found the problem in "Elementary Calculus and Coordinate Geometry" by Nobbs[2] but in the hands of a skilful storyteller the problem can be worded to sound like a "real-life" problem.

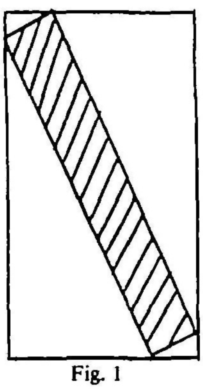

Fig. 1

A suitable sketch of the problem usually looks like that in Fig. 1. At this stage there tend to be three main lines of attack proposed – these are, briefly:

(*i*) accurate drawings and/or physical model making,

(*ii*) geometric intuitionism, and

(*iii*) symbol bashing.

Taking the last of these first we now need to introduce symbols to represent the important variables. A possible set of symbols is shown in Fig. 2.

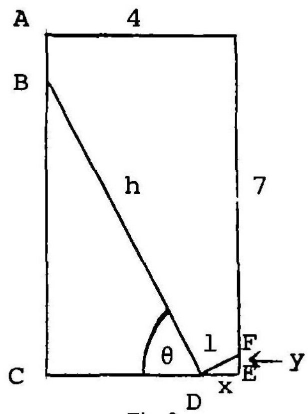

Fig. 2

3. A model

Since the object of the problem is to find the maximum value of h it would seem reasonable to find h as a function of one of the other variables and to differentiate. Suppose we try θ first, then we can find an expression involving h and θ by considering the width of the door. Since BDF is a right-angle then angle $DFE = \theta$, and since $CD + DE = CE$ we have:

$$h \cos\theta + \sin\theta = 4. \tag{1}$$

Re-arranging yields

$$h = 4\sec\theta - \tan\theta, \tag{2}$$

and differentiating produces

$$\frac{dh}{d\theta} = 4\sec\theta\tan\theta - \sec^2\theta.$$

Setting this equal to zero gives

$$4\tan\theta = \sec\theta, \text{ assuming } \sec\theta \neq 0$$

and thus

$$\sin\theta = \frac{1}{4}, \cos\theta = \frac{\sqrt{15}}{4}$$

and

$$\tan\theta = \frac{1}{\sqrt{15}}, \text{ i.e., } \theta \doteqdot 14°.$$

Substituting in (2) yields

$$h = \frac{16}{\sqrt{15}} - \frac{1}{\sqrt{15}} = \frac{15}{\sqrt{15}} = \sqrt{15}.$$

Since $\sqrt{15}$ is a shade less than 4 we could expect that it would be noticed that the solution is not realistic. We could now try to find another method, or perhaps explore what this solution means.

First let us sort out what is wrong with *this* model, and with any luck this might improve our model. Working out $\frac{d^2h}{d\theta^2}$ very quickly shows that our value $h = \sqrt{15}$ represents a *minimum*, so what problem were we actually solving? Another look at Fig. 2 shows that we did not stipulate that all four corners of the bookcase should touch the door frame. What we found above is the minimum length of bookcase of which three corners (*i.e., B, D, F*) will touch.

4. A refined model

Ignoring θ this time we could use Pythagoras in triangles *BCD* and *DEF* to find relationships connecting h, x and y

i.e.,
$$h^2 = (4-x)^2 + (7-y)^2, \tag{3}$$
and
$$x^2 + y^2 = 1. \tag{4}$$
Eliminating y yields
$$h^2 = (4-x)^2 + (7-\sqrt{1-x})^2. \tag{5}$$

At this stage we have achieved h^2 as a function of x. A reasonable argument would be that the value of x that maximises h will also maximise h^2 so the expression could be differentiated as it stands.

$$\frac{d(h^2)}{dx} = -2(4-x) + \frac{2x(7-\sqrt{1-x^2})}{\sqrt{1-x^2}}.$$

Setting this equal to zero gives

$$x(7-\sqrt{1-x^2}) = (4-x)\sqrt{1-x^2},$$

which on squaring and simplifying gives $65x^2 = 16$.

Putting $x = \frac{4}{\sqrt{65}}$ in (5) produces the approximate solution $h = 7.06$ft.

This, then, is a much more plausible solution and we could regard the problem as duly solved. Does it agree with the findings of the other approaches?

5. Geometric intuition

If the doorway had been square we would expect triangle *DEF* to be isosceles. If we consider the doorway distorting we could expect the triangle *DEF* somehow to distort similarly. We could suppose that if the sides of the doorway are in the ratio 4:7 then the sides of the triangle would be in the same ratio. Since $4^2 + 7^2 = 65$, this would correspond to $x = \frac{4}{\sqrt{65}}$ and $y = \frac{7}{\sqrt{65}}$. Substituting these in (3) gives precisely the same solution of $h = 7.06$ as obtained by the calculus route. This, then, would seem to be proof positive – intuition confirmed by analysis.

6. An accurate drawing

This might seem redundant by now but it certainly would do no harm to use this to confirm the findings of the other methods. We just need a scale drawing of a 4 by 7 rectangle in which we inscribe a rectangle in such a way that x represents $\frac{4}{\sqrt{65}}$ and y represents $\frac{7}{\sqrt{65}}$, and measure the sides to be 1 and 7.06. Can you draw it?

Even if the draughtsmen cannot produce a solution by themselves they should be able to validate a solution found by other means. At this stage you should break off reading and try validating the solution for yourself!

7. Error detection and correction

If the accurate drawing fails to confirm the proposed solution then where did we go wrong? Assuming the differentiation was performed correctly then perhaps we should back-track to the

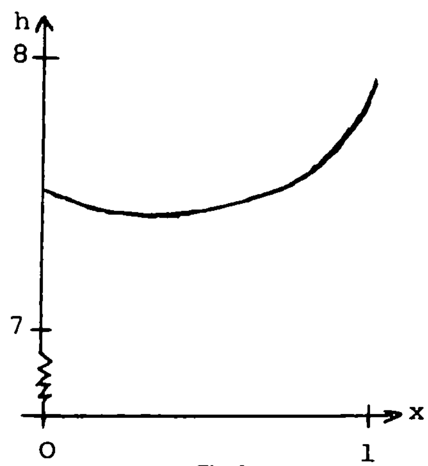

Fig. 3

function in (5). Since x is the displacement of the bottom corner of the bookshelf horizontally from the corner of the doorframe then x must lie between 0 and 1, so we could sketch a graph of h against x for this range of x as in Fig. 3. So our solution was a *minimum* of some kind – but what problem have we modelled? The element of the physical problem that we have lost is the right-angledness of BDF, so we are considering different ways of inserting a parallelogram into the rectangular aperture with sides of length $l$ft and $h$ft. Our solution is the shortest parallelogram cross-section bookshelf that will just touch the edges of the aperture!

8. Modelling the missing element

Returning to the original model in (1) we could derive a similar expression for the height of the door using $BC + AB = AC$, *i.e.*,

$$h \sin \theta + \cos \theta = 7, \tag{6}$$

which together with equation (1)

$$h \cos \theta + \sin \theta = 4, \tag{1}$$

yields a pair of nonlinear simultaneous equations in h and θ. Eliminating h between the equations yields an equation for θ

$$4 \sin \theta - \sin^2 \theta = 7 \cos \theta - \cos^2 \theta, \tag{7}$$

and eliminating $\tan \theta$ between them yields an equation in h

$$h^4 - 67h^2 + 112h - 64 = 0. \tag{8}$$

Using an appropriate numerical technique we could solve either equation, in which case we obtain an approximate solution of $h = 7.25$ft when $\theta = 65°$.

9. Postscript

At the heart of this tale of red herrings lies the often used word "maximise." Where, then, did the maximisation take place if the solution just came from the solution of equations? Here the answer is so obvious that perhaps it is not surprising that we overlooked it – just by assuming that the corners of the bookshelf touch the wall! If it is consolation to anyone it might be of interest to note that our PGCE mathematics graduates are just as prone to fall for the red herrings as our BEd post A-level students.

References

1. Raggett, G. F., *Teaching Mathematics and its Applic.*, 1982, **1**, 31.
2. Nobbs, G. C., "Elementary Calculus and Coordinate Geometry," Part II, OUP, 1949, p. 363.

Adrian Oldknow received his mathematics degree from Oxford University.

After teaching mathematics in schools he moved to a lectureship in mathematics and computing at Reading College of Technology. He took an MTech degree in computer science and numerical analysis at Brunel University. He is currently a principal lecturer in mathematics and computing at the West Sussex Institute of Higher Education which has its mathematics centre in Bognor Regis. He is a committee member of the Mathematics Applicable Group and his main interests are in the applications of mathematics particularly to interactive computer graphics having recently spent a sabbatical period at the Computer Aided Design Centre in Cambridge.